INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 13, 579–597 (1991)

STABILITY ANALYSIS OF FINITE DIFFERENCE SCHEMES FOR TWO-DIMENSIONAL ADVECTION–DIFFUSION PROBLEMS

ALAIN RIGAL

Laboratoire d'Analyse Numérique, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex, France

SUMMARY

This paper develops a stability analysis of second-order, two- and three-time-level difference schemes for the 2D linear diffusion–convection model problem. The corresponding 1D schemes have been extensively analysed in two previous papers by the same author. Most of these 2D schemes obviously generalize 1D schemes, i.e. their stencil only uses the nearest points and defines 'product difference schemes'; however, the stability results are not always the exact generalization of the 1D stability properties. Moreover, the 1D non-viscous MFTCS scheme may only be generalized if one uses a nine-point scheme. Numerical experiments for different values of the cell Reynolds number allow a comparison to be made between the theoretical and numerical stability limits.

KEY WORDS Diffusion–convection Fourier analysis Stability Artificial viscosity

1. INTRODUCTION

In two previous papers[1,2] we analysed the properties of second-order finite difference schemes for the one-dimensional advection–diffusion problem. These papers, devoted to two- and three-time-level schemes respectively, will be referred to hereafter as R1 and R2. In the present paper we study the stability of the schemes analysed in R1 and R2 when applied to multidimensional advection–diffusion problems.

Many authors[3–11] have developed stability analyses of different two- and three-level schemes. Thus several results in this paper, in particular for basic schemes, are not new. They are, however, detailed because our approach provides complementary information. Moreover, the stability of the fundamental FTCS (forward time, centred space) scheme was erroneously analysed by Fromm in 1964.[12] This incorrect result was promulgated by Roache,[13] but was corrected several years later.[3,5,6] Our previous stability result relative to the FTCS scheme was only correct for 2D isotropic problems;[11] as proved in R1, the restriction of this result to the 1D problem is exact. Among the papers quoted above, the recent work of Hindmarsh *et al.*[3] is by far the most complete.

The analysis of the schemes will be described in detail for 2D problems and may be generalized without difficulty to multidimensional problems (in $\mathbb{R}^n$, $n \geqslant 3$); indeed, the crucial differences appear between 1D (R1 and R2) and 2D problems.

0271–2091/91/150579–19$09.50

Received January 1990
Revised October 1990

We consider the linear homogeneous advection–diffusion problem

$\partial_t u = Au$ in $\Omega \times]0, T[$, Ω a bounded domain of $\mathbb{R}^2$, with initial and boundary conditions, (P)

i.e.

$$\begin{aligned} \partial_t u + c_1 \partial_x u + c_2 \partial_y u &= K_1 \partial_x^2 u + K_2 \partial_y^2 u, \\ u = h \quad &\text{on} \quad \partial\Omega \times]0, T[, \\ u(x, y, 0) = u_0(x, y) \quad &\text{on } \Omega, \end{aligned} \tag{1}$$

where the c_i are the advection velocities and the K_i are the diffusion coefficients.

Firstly, let us recall the main classes of schemes analysed in our previous papers (R1 and R2) when Ω is a bounded interval of $\mathbb{R}$.

The two-level schemes analysed in R1 may be written

$$(I - \theta \Delta t A_h) V^{n+1} = [I + (1-\theta)\Delta t A_h] V^n, \tag{2}$$

where A_h is a space difference operator which realizes an approximation of order at most two of the diffusion–convection operator A, and θ belongs to $[0, 1]$ ($\theta = 0$, explicit schemes).

In R2, two classes of three-level schemes are considered:

(a) Weighted (W) schemes deduced from classical two-step schemes used for differential equations
(b) Leap-frog (LF) schemes using space- and time-centred differencing.

The W-schemes are defined by

$$(\theta + \tfrac{1}{2}) v_j^{n+1} - 2\theta v_j^n + (\theta - \tfrac{1}{2}) v_j^{n-1} = \Delta t A_h \bar{v}_j, \quad \theta \in [0, 1], \tag{3}$$

where $\bar{v}_j$ is a linear combination of v_j^n, v_j^{n+1} and v_j^{n-1} and A_h is defined as above in (2).

The LF schemes are defined by

$$\frac{v_j^{n+1} - v_j^{n-1}}{2\Delta t} = B_h \bar{v}_j - c \frac{v_{j+1}^n - v_{j-1}^n}{2h}, \tag{4}$$

where $\bar{v}_j$ is as defined above and B_h is a space difference operator yielding second-order approximations of the diffusion term.

The leap-frog Du Fort–Frankel (LFDF) scheme

$$\frac{v_j^{n+1} - v_j^{n-1}}{2\Delta t} = \frac{v_{j+1}^n + v_{j-1}^n - v_j^{n+1} - v_j^{n-1}}{h^2} - c \frac{v_{j+1}^n - v_{j-1}^n}{2h} \tag{5}$$

cannot be written as (4).

The generalization of schemes (2)–(5) to multidimensional problems is straightforward if we consider second-order space operators using five points in $\mathbb{R}^2$ or seven points in $\mathbb{R}^3$ ('product schemes' of 1D schemes).

2. TWO-LEVEL SCHEMES

We first consider explicit schemes which allow us to focus on the properties of space differencing. Moreover, R1 showed the interesting properties of explicit schemes for solving strongly convective problems.

2.1. *Explicit schemes*

We consider

$$V^{n+1} = (I + \Delta t A_{ph}) V^n, \tag{6}$$

with the following definitions of A_{ph}, $p=1, 2, 3$:

$$A_{1h}=\left(K_1+c_1^2\frac{\Delta t}{2}\right)D_x^+D_x^-+\left(K_2+c_2^2\frac{\Delta t}{2}\right)D_y^+D_y^--c_1D_{0x}-c_2D_{0y}, \tag{7}$$

modified FTCS scheme (MFTCS) analysed in the paper of Hindmarsh *et al.*;[3]

$$A_{2h}=K_1D_x^+D_x^--c_1\delta_1D_x^--c_1(1-\delta_1)D_{0x}+K_2D_y^+D_y^--c_2\delta_2D_y^--c_2(1-\delta_2)D_{0y}, \tag{8}$$

weighted upstream scheme (FTUW) which gives the FTCS scheme when $\delta_1=\delta_2=0$ and the FTUS scheme (basic upstream scheme) when $\delta_1=\delta_2=1$;

$$A_{3h}=\frac{K_1}{1+c_1h_1/2K_1}D_x^+D_x^-+\frac{K_2}{1+c_2h_2/2K_2}D_y^+D_y^--c_1D_{0x}-c_2D_{0y}, \tag{9}$$

Samarskii scheme[14] including an *a priori* correction of the artificial viscosity; where D_x^+, D_x^- and D_{0x} are the forward, backward and centred difference operators following the x-axis, with a constant space step h_1.

Remark. In contrast to the MFTCS scheme studied in R1, the modified scheme (7) does not correct the artificial viscosity; indeed, in the artificial viscosity of the 2D FTCS scheme a cross-derivative $\partial^2_{xy}u$ appears which cannot be equalized with difference operators when using a five-point scheme. Dukawicz and Ramshaw,[15] who use finite differences, and Hindmarsh *et al.*,[3] who use finite elements, proposed schemes which take this observation into account. In an analogous way we define a nine-point scheme, MFTCD (cross-derivative), which we will analyse in a further paragraph.

The stability of the difference scheme (6) applied to the Cauchy problem associated with (P) results from the analysis of the amplification factor

$$g_p(\varphi_1, \varphi_2)=1-Z_p, \quad \text{with } \varphi_i=\omega_ih_i \quad \text{in } [0, \pi], \tag{10}$$

$$Z_p=2d_{p_1}(1-\cos\varphi_1)+2d_{p_2}(1-\cos\varphi_2)+\mathrm{i}(\mu_1\sin\varphi_1+\mu_2\sin\varphi_2) \tag{11}$$

where the d_{p_i} depend on the A_{ph} in (7)–(9). In terms of the characteristic parameters

$$r_i=K_i\frac{\Delta t}{h_i^2}, \quad \text{parabolic mesh ratio,}$$

$$\mu_i=c_i\frac{\Delta t}{h_i}, \quad \text{hyperbolic mesh ratio,}$$

$$\alpha_i=\frac{c_ih_i}{2K_i}, \quad \text{cell Reynolds number,}$$

the coefficients d_{p_i} are defined by

$$d_{1i}=r_i+\frac{\mu_i^2}{2} \qquad \text{for } A_{1h}, \tag{12a}$$

$$d_{2i}=r_i+\frac{\mu_i\delta_i}{2} \qquad \text{for } A_{2h}, \tag{12b}$$

$$d_{3i}=r_i(1+\alpha_i)^{-1}+\frac{\mu_i}{2} \quad \text{for } A_{3h}. \tag{12c}$$

Prescribing the stability condition

$$\sup_{(\varphi_1, \varphi_2)\in[0,\pi]^2} |g_p(\varphi_1, \varphi_2)| \leqslant 1$$

we obtain the following theorem.

Theorem 1

The explicit scheme (6) is stable if

$$d_{p_1} + d_{p_2} \leqslant \tfrac{1}{2}, \tag{13}$$

$$\frac{\mu_1^2}{2d_{p_1}} + \frac{\mu_2^2}{2d_{p_2}} < 1. \tag{14}$$

Proof. We drop the index p and study the extrema of

$$G(\varphi_1, \varphi_2) = |g(\varphi_1, \varphi_2)|^2 = 1 - 4d_1(1-\cos\varphi_1) - 4d_2(1-\cos\varphi_2) + 4d_1^2(1-\cos\varphi_1)^2 + 4d_2^2(1-\cos\varphi_2)^2 + 8d_1 d_2(1-\cos\varphi_1)(1-\cos\varphi_2) + (\mu_1 \sin\varphi_1 + \mu_2 \sin\varphi_2)^2 \tag{15}$$

over $[0, \pi] \times [0, \pi]$. From the first derivatives of G we deduce that

(i) $(0, 0)$, $(0, \pi)$, $(\pi, 0)$ and (π, π) are stationary points for G
(ii) a stationary point of G within the square $]0, \pi[\times]0, \pi[$ belongs to the curve defined by

$$\frac{d_1}{\mu_1} \tan\varphi_1 = \frac{d_2}{\mu_2} \tan\varphi_2. \tag{16}$$

These stationary points must be maxima for G and the values taken by G must be bounded by unity.

(i) In any case, $G(0, 0) = 1$ (consistency) and $(0, 0)$ is a maximum for G if

$$[G''_{\varphi_1\varphi_2}(0, 0)]^2 - G''_{\varphi_1^2}(0, 0)\, G''_{\varphi_2^2}(0, 0) < 0,$$

$$G''_{\varphi_1^2}(0, 0) < 0, \qquad G''_{\varphi_2^2}(0, 0) < 0,$$

which are satisfied if

$$\frac{\mu_1^2}{2d_1} + \frac{\mu_2^2}{2d_2} < 1, \quad \text{i.e. (14);}$$

moreover, $G(\pi, \pi) \leqslant 1$ yields

$$d_1 + d_2 \leqslant \tfrac{1}{2}, \quad \text{i.e. (13),}$$

and (π, π) is a maximum for G when (13) and (14) are satisfied. In the same way, $G(0, \pi) = G(\pi, 0) \leqslant 1$ if (13) and (14) are satisfied.

(ii) Within $]0, \pi[\times]0, \pi[$, the curve defined by (16) can be described parametrically as

$$\tan\varphi_1 = \frac{\mu_1}{d_1} u, \qquad \tan\varphi_2 = \frac{\mu_2}{d_2} u,$$

and we have to minimize $\tilde{G}(u) = G(\varphi_1, \varphi_2)$. A similar analysis made by Cushman-Roisin[16] for the LFDF scheme exhibits maxima for $\tilde{G}$ when $(\varphi_1, \varphi_2) \to (0, 0)$ and $(\varphi_1, \varphi_2) \to (\pi, \pi)$. Thus we return to the previous case (i) and conclude that (13) and (14) are the stability conditions of the explicit scheme (6). □

Geometrical interpretation

The amplification factor $g_p(\varphi_1, \varphi_2)$ may be written as

$$g_p(\varphi_1, \varphi_2) = z_1 + z_2,$$

with

$$z_i = \tfrac{1}{2} - 2d_{p_i} + 2d_{p_i} \cos\varphi_i - \mathrm{i}\mu_i \sin\varphi_i.$$

In the complex plane, when φ_i belongs to $[0, \pi]$, z_i describes the half-ellipse (E_i) with axes $(2d_{p_i}, \mu_i)$ and centre C_i $(\tfrac{1}{2} - 2d_{p_i}, 0)$. The above conditions, (13) for $(\varphi_1, \varphi_2) = (\pi, \pi)$ and (14) for $(\varphi_1, \varphi_2) = (0, 0)$, present a clear geometrical support. Indeed (Figure 1), (13) expresses that the sum of the moduli of z_i is lower than unity towards $x < 0$, and (14) expresses that the sum of the radii of curvature of the ellipses at point A $(\tfrac{1}{2}, 0)$ is lower than unity:

$$R_i = \frac{(x_i'^2 + y_i'^2)^{3/2}}{y_i'' x_i' - x_i'' y_i'} = \frac{(4d_{p_i}^2 \sin^2\varphi_i + \mu_i^2 \cos^2\varphi_i)^{3/2}}{2d_{p_i}\mu_i \sin^2\varphi_i}$$

are such that

$$R_1 + R_2 \to \frac{\mu_1^2}{2d_{p_1}} + \frac{\mu_2^2}{2d_{p_2}} \quad \text{when } (\varphi_1, \varphi_2) \to (0, 0).$$

Therefore the stability condition (14) is equivalent to

$$R_1 + R_2 < 1 \quad \text{at } (0, 0).$$

This geometrical interpretation confirms that the stability criterion

$$G(\varphi_1, \varphi_2) \leqslant 1 \quad \text{for} \quad (\varphi_1, \varphi_2) \in [0, \pi]^2 \tag{17}$$

is satisfied if (17) is valid near (0, 0) and (π, π).

Moreover, we derive an important practical conclusion: Figure 1 clearly shows that the behaviour of $G(\varphi_1, \varphi_2)$ is very different according to whether (φ_1, φ_2) tends to (0, 0) or to (π, π); near (0, 0), $G(\varphi_1, \varphi_2)$ may only slightly overlap unity (A always belongs to both ellipses), while if

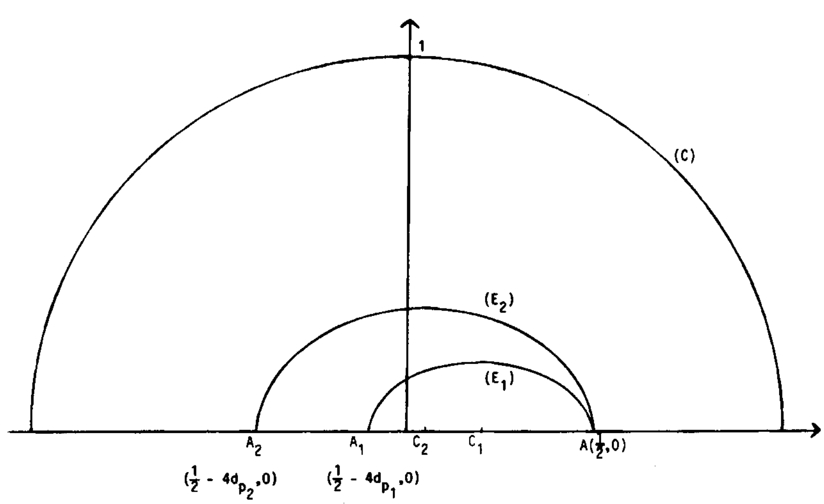

Figure 1. Ellipses which support the geometrical interpretation

$G(\pi, \pi)$ is greater than unity, we have $G(\varphi_1, \varphi_2) > 1$ in certain intervals of frequency $[\tilde{\varphi}_1, \pi]$ and $[\tilde{\varphi}_2, \pi]$. Therefore the condition on the time step resulting from (13) has to be strictly satisfied, whereas the condition resulting from (14) may be significantly overlapped.

Corollary

For the isotropic problem ($c_1 = c_2 = c$, $K_1 = K_2 = K$) and a regular mesh ($h_1 = h_2 = h$) the stability conditions (13) and (14) become

$$r \leqslant \left\{ r_{11} = \frac{1}{2\alpha^2}, \quad r_{12} = \frac{1}{2[1 + \sqrt{(1 + 2\alpha^2)}]} \right\} \quad \text{(MFTCS scheme),} \tag{18}$$

$$r \leqslant \left\{ r_{21} = \frac{1 + \alpha\delta}{4\alpha^2}, \quad r_{22} = \frac{1}{4(1 + \alpha\delta)} \right\} \quad \text{(FTUW scheme),} \tag{19}$$

$$r \leqslant r_{32} = \frac{1 + \alpha}{4(1 + \alpha + \alpha^2)} \quad \text{(Samarskii scheme).} \tag{20}$$

The bounds r_{i1} and r_{i2}, which derive from (14) and (13) respectively, are given in terms of the cell Reynolds number $\alpha = ch/2K$. For the Samarskii scheme, $r_{31} > r_{32}$ for any α.

Conditions (18), (19) and (20) must be compared with (27), (34) and (40) respectively in R1. The main difference concerns the MFTCS scheme, which does not generalize the 1D MFTCS scheme analysed in R1: the behaviour of the stability limit is α^{-2} for large values of α as opposed to α^{-1} in the one-dimensional case. Figures 2–5 exhibit the theoretical and experimental stability limits of the above schemes plotted on the same scale. In these figures we observe that the stability properties of the schemes are almost identical for small values of α. For the most interesting problems (i.e. strongly convective problems, $\alpha \geqslant 2$) the theoretical and especially practical stability conditions of the Samarskii scheme are the most favourable.

Remark. The bounds r_{i2} generally have to be strictly verified; however, this requirement is less rigorous when α increases. The bounds r_{i1}, which only occur when α is large, may be overlapped; the schemes usually remain stable when $r \approx 2r_{i1}$. These different types of behaviour are somehow

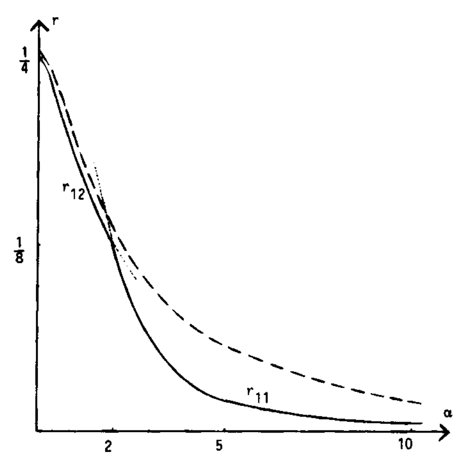

Figure 2. MFTCS scheme: theoretical stability limits ($r_{12}(\alpha)$ when $\alpha \leqslant 2$, $r_{11}(\alpha)$ when $\alpha \geqslant 2$) and experimental stability limit (broken line)

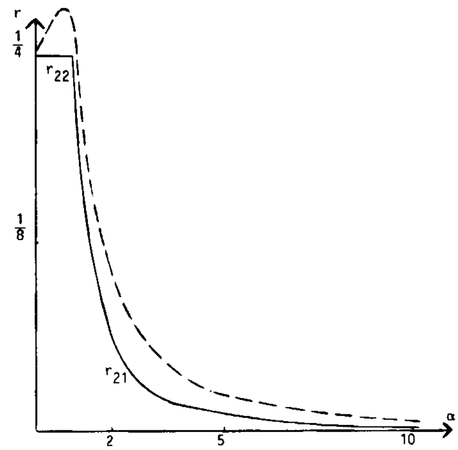

Figure 3. FTCS scheme (i.e. FTUW scheme with $\delta=0$): theoretical stability limits ($r_{22}(\alpha)$ when $\alpha \leqslant 1$, $r_{21}(\alpha)$ when $\alpha > 1$) and experimental stability limit (broken line)

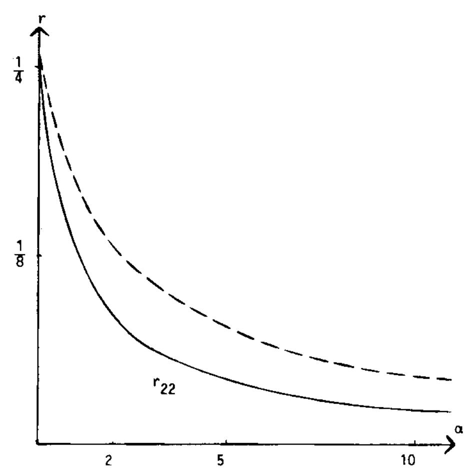

Figure 4. FTUS scheme (i.e. FTUW scheme with $\delta=1$): theoretical stability limit ($r_{22}(\alpha)$) and experimental stability limit (broken line)

attenuated because they correspond to unstable modes at the ends of the frequency spectrum (near zero or π). When unstable modes correspond to frequencies within $]0, \pi[$ (as for some three-level schemes; see Section 3), the stability condition must be rigorously verified.

Finally, we must specify that the accuracy of the schemes is not considered: there is no comparison between different numerical results. In particular, near the experimental stability limit the numerical solutions are frequently oscillatory before they attain a stable steady state solution.[17]

2.2. Implicit schemes

The weighted implicit scheme associated with (6) is defined by

$$(I-\theta\Delta t A_{ph})V^{n+1}=[I+(1-\theta)\Delta t A_{ph}]V^{n}, \quad \theta \in]0, 1], \tag{21}$$

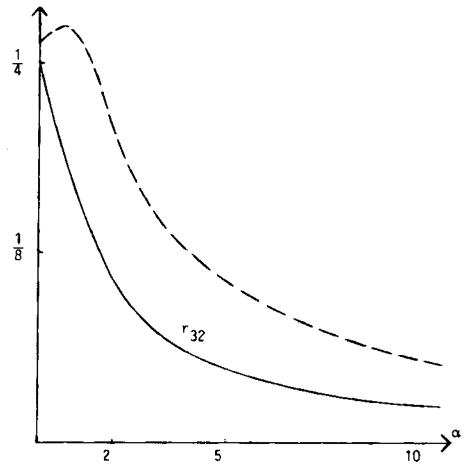

Figure 5. Samarskii scheme: theoretical stability limit ($r_{32}(\alpha)$) and experimental stability limit (broken line)

where A_{2h} and A_{3h} are given by (8) and (9) and A_{1h} depends on θ:

$$A_{1h}=\left(K_1+c_1^2\frac{\Delta t}{2}(1-2\theta)\right)D_x^+D_x^- -c_1D_{0x}+\left(K_2+c_2^2\frac{\Delta t}{2}(1-2\theta)\right)D_y^+D_y^- -c_2D_{0y}. \quad (22)$$

The amplification factor relative to (21) is written as

$$\gamma_p(\varphi_1,\varphi_2)=1-\frac{Z_p}{1+\theta Z_p},\quad Z_p \text{ given by (11).} \quad (23)$$

We prescribe $|\gamma_p(\varphi_1,\varphi_2)|\leqslant 1$ over $[0,\pi]\times[0,\pi]$ and obtain the following theorem.

Theorem 2

When $\theta\geqslant 0{\cdot}5$, the implicit scheme (21) is stable for all Δt and h. When $\theta<0{\cdot}5$, the scheme is stable if

$$d_{p_1}+d_{p_2}\leqslant\frac{1}{2(1-2\theta)}, \quad (24)$$

$$\frac{\mu_1^2}{2d_{p_1}}+\frac{\mu_2^2}{2d_{p_2}}<\frac{1}{1-2\theta}. \quad (25)$$

Proof. Taking the modulus of the amplification factor, we obtain

$$|\gamma_p|^2=1+\frac{R}{D^2},$$

with

$$R=4(1-2\theta)[d_{p_1}(1-\cos\varphi_1)+d_{p_2}(1-\cos\varphi_2)]^2-4[d_{p_1}(1-\cos\varphi_1)+d_{p_2}(1-\cos\varphi_2)]$$
$$+(1-2\theta)(\mu_1\sin\varphi_1+\mu_2\sin\varphi_2)^2,$$

which must be negative. When $\theta\geqslant 0{\cdot}5$, R is always negative; when $\theta<0{\cdot}5$, we replace $R<0$ by the equivalent inequality

$$(1-2\theta)R+1<1,$$

i.e.

$$\{1-2(1-2\theta)[d_{p_1}(1-\cos\varphi_1)+d_{p_2}(1-\cos\varphi_2)]\}^2+[(1-2\theta)(\mu_1\sin\varphi_1+\mu_2\sin\varphi_2)]^2<1. \quad (26)$$

Inequality (26) is identical to condition (15) expressing the stability of the explicit schemes, except that d_i and μ_i are replaced by $d_{p_i}(1-2\theta)$ and $\mu_i(1-2\theta)$ respectively. Therefore we deduce the stability conditions (24) and (25) which generalize (13) and (14) for $\theta\in]0, 0{\cdot}5[$. □

2.3. Nine-point non-viscous scheme

A_{1h} in (7) defines the MFTCS scheme which generalizes the one-dimensional MFTCS scheme analysed in R1. However, in contrast to the 1D scheme, A_{1h} does not completely eliminate the artificial viscosity produced by the basic centred (FTCS) scheme. Indeed, in the 1D case the artificial viscosity vanishes using a Lax–Wendroff correction based on the relation between $\partial_t^2 u$ and the space derivatives for $u(x, t)$, the solution of (P).

For 2D problems this relation becomes

$$\partial_t^2 u=c_1^2\partial_x^2 u+c_2^2\partial_y^2 u+2c_1c_2\partial_{xy}^2 u+\text{HOD}, \quad (27)$$

where HOD represents higher-order derivatives (of order greater than two), and the correction of the diffusion terms in (7) does not take into account the cross-derivative $\partial_{xy}^2 u$.

Thus we define a nine-point scheme where the second-order difference operators are defined so as to balance the second-order derivatives in (27):

$$A_{4h}=\left(K_1+c_1^2\frac{\Delta t}{2}\right)D_x^+D_x^- - c_1D_{0x}+\left(K_2+c_2^2\frac{\Delta t}{2}\right)D_y^+D_y^- - c_2D_{0y}+c_1c_2\Delta t D_{0x}D_{0y}. \quad (28)$$

This scheme will be denoted as the MFTCD (cross-derivative) scheme and may be considered as a particular case of (100) in Reference 3. For the corresponding weighted scheme (θ-MCD scheme)

$$(I-\theta\Delta t A_{4h})V^{n+1}=[I+(1-\theta)\Delta t A_{4h}]V^n, \quad (29)$$

we obtain the following partial stability results (d_{1i} given by (12a)).

Theorem 3

When $\theta\geqslant 0{\cdot}5$, the nine-point scheme (29) is stable for all Δt and h. When $\theta<0{\cdot}5$, a necessary stability condition is

$$d_{11}+d_{12}\leqslant\frac{1}{2(1-2\theta)}. \quad (30)$$

Proof. The amplification factor associated with (29) is given by

$$g_4(\varphi_1, \varphi_2)=\frac{1-(1-\theta)Z_4}{1+\theta Z_4},$$

with

$$Z_4=2d_{11}(1-\cos\varphi_1)+2d_{12}(1-\cos\varphi_2)+\mu_1\mu_2\sin\varphi_1\sin\varphi_2+\mathrm{i}\mu_1\sin\varphi_1+\mathrm{i}\mu_2\sin\varphi_2,$$

$$d_{1i}=r_i+\frac{\mu_i^2}{2}.$$

The analysis of $|g_4|$ is closely related to previous stability analyses:

(a) Near (0, 0), $|g_4(\varphi_1, \varphi_2)| \leqslant 1$ for any value of θ; thus we do not have a stability condition similar to (25).
(b) Near (π, π), we obtain (30) which is identical to stability condition (24) obtained for the five-point schemes.

The study of $|g_4(\varphi_1, \varphi_2)|$ in $]0, \pi[\times]0, \pi[$ is extremely involved. We observe that this modulus may be a maximum in $]0, \pi[\times]0, \pi[$, but it is not possible to deduce simple sufficient stability conditions. □

However, for isotropic problems and a regular grid we may obtain satisfactory sufficient stability conditions.

Let us consider the explicit (MFTCD) scheme (the results for the θ-scheme are then straightforward). We can easily observe that extrema for $|g_4|$ occur when $\varphi_1 = \varphi_2 = \varphi$. Thus the MFTCD scheme will be stable if

$$|g| = |1 - 4d(1 - \cos\varphi) - \mu^2 \sin^2\varphi - 2i\mu \sin\varphi| \leqslant 1, \quad \text{when } \varphi \text{ belongs to }]0, \pi[,$$

i.e.

$$|g|^2 = 64d^2 X + 16\mu^4 X(1-X)^2 + 8\mu^2(1-X)X + 64\mu^2 dX^2(1-X) - 16dX + 1 \leqslant 1, \tag{31}$$

where

$$X = \sin^2\frac{\varphi}{2} \quad \text{in }]0, 1[.$$

We obtain the following theorem.

Theorem 4

The MFTCD scheme is stable if

$$r < r_s = \inf\{r_1, r_2\},$$

with

$$r_1 = \frac{1}{2[1 + \sqrt{(1 + 2\alpha^2)}]}, \qquad r_2 = \frac{1}{2\alpha\sqrt{6}}. \tag{32}$$

Remark. $r < r_1$ is the condition deduced from the necessary condition (30) in the isotropic case.

Proof. Inequality (31) yields the following condition for the cubic polynomial $P(X)$:

$$P(X) = 2\mu^4 X^3 - 8\mu^2(\mu^2 + r)X^2 + [8(r + \mu^2)^2 - \mu^2]X - 2r \leqslant 0, \quad \text{whatever } X \text{ in }]0, 1[. \tag{33}$$

Note that $P(0)$ is always negative and $P(1) < 0$ corresponds to the necessary stability condition (30).

Condition (33) is in particular satisfied if P does not present a maximum in $]0, 1[$. $P(X)$ presents at most one extremum in $[0, 1]$ when X takes the value

$$X_m = \frac{4(\mu^2 + r) - [4(\mu^2 + r)^2 + 3\mu^2/2]^{1/2}}{3\mu^2}. \tag{34}$$

From $P'(X)$ we deduce the following.

(a) $P'(0)<0$ if

$$r<r^*=\frac{\alpha-\sqrt{2}}{4\alpha^2\sqrt{2}} \qquad (\text{and } \alpha>\sqrt{2}). \tag{35}$$

(b) $P'(0)$ and $P'(1)$ are positive if

$$r<r^{**}=\frac{\sqrt{(2-\alpha^2)}}{2\alpha^2\sqrt{2}} \qquad (\text{and } \alpha<\sqrt{2}). \tag{36}$$

In both cases $P(X)$ does not present a maximum and the scheme is stable.

When $\alpha<\alpha_1=\sqrt{\frac{3}{2}}$, we have

$$r_1<r^{**};$$

the necessary condition $r<r_1$ (see remark above) is then sufficient.

Now, when neither (35) nor (36) is satisfied, i.e. if

$$\alpha\in[\sqrt{\tfrac{3}{2}},\ \sqrt{2}], \qquad r\in[r^{**},\ r_1]$$

or

$$\alpha>\sqrt{2}, \qquad r\in[r^*,\ r_1],$$

we prescribe $P(X_{\mathrm{m}})<0$. Replacing $\mu=2\alpha r$, we obtain the following condition for the quartic polynomial $Q(r)$:

$$Q(r)=-2048\alpha^8r^4-2560\alpha^6r^3+128\alpha^4(\alpha^2-9)r^2+16\alpha^2(13\alpha^2-14)r-2\alpha^4+71\alpha^2-16<0. \tag{37}$$

This condition is satisfied for $r=r^{**}$ and $r=r^*$. Thus the first root of Q greater than r^{**} or r^* yields a stability bound r_{s} for r.

This bound may be obtained

(i) either by computing (for given α) an approximate value from the starting value $r^{(0)}=r^*$ or r^{**} (following α)

(ii) or by defining the smallest interval with centre r^* or r^{**} which does not contain any root of Q.

For (ii), Henrici[18] gives different estimates based on the local development of Q. We must calculate the coefficients

$$b_m=\frac{1}{m!}Q^{(m)}(r^{**})$$

and study

$$\min_{m=1,4}\left|\frac{b_0}{b_m}\right|^{1/m}.$$

This analysis is rather involved and does not generally provide a precise estimate for r_{s}.

From this remark we infer that it would be preferable to obtain a constant C such that

$$Q\left(\frac{C}{\alpha}\right)>0 \quad \text{for } \alpha>\alpha_1.$$

With $C=1/2\sqrt{6}$ we have

$$Q\left(\frac{C}{\alpha_1}\right)=0 \quad \text{and} \quad Q\left(\frac{C}{\alpha}\right)>0 \quad \text{on }]\alpha_1,\ \alpha_2] \quad \text{with } \alpha_2>90.$$

Therefore $r<r_2=1/2\alpha\sqrt{6}$ is a sufficient stability condition in the useful area for the cell Reynolds number α. □

Figure 6 shows the curves r_1, r_2, r^{**} and r^* in the (α, r)-plane and the experimental stability limit. We may observe that the sufficient conditions (heavy line) of Theorem 6 are quite realistic.

Note that the θ-MCD scheme is stable when $\theta>0{\cdot}5$ and if

$$r\leqslant\frac{r_s}{1-2\theta},\quad r_s \text{ given by (32)},\quad \text{when } \theta<0{\cdot}5. \tag{38}$$

The MFTCD scheme generalizes the 1D MFTCS scheme analysed in R1; its main (and interesting) characteristic is the absence of artificial viscosity. This scheme does not present a stability domain larger than those of five-point schemes previously analysed. Moreover, the matrix properties of nine-point schemes are less favourable (sparsity, monotonicity, etc.).

The present incomplete analysis—excluding accuracy parabolicity and so on—does not generally justify the utilization of the second-order nine-point MFTCD scheme.

3. THREE-LEVEL SCHEMES

In R2 we showed that three-level schemes did not offer significant advantages over the two-level schemes studied in R1, particularly for strongly convective problems. Thus we will not give a detailed analysis of these schemes.

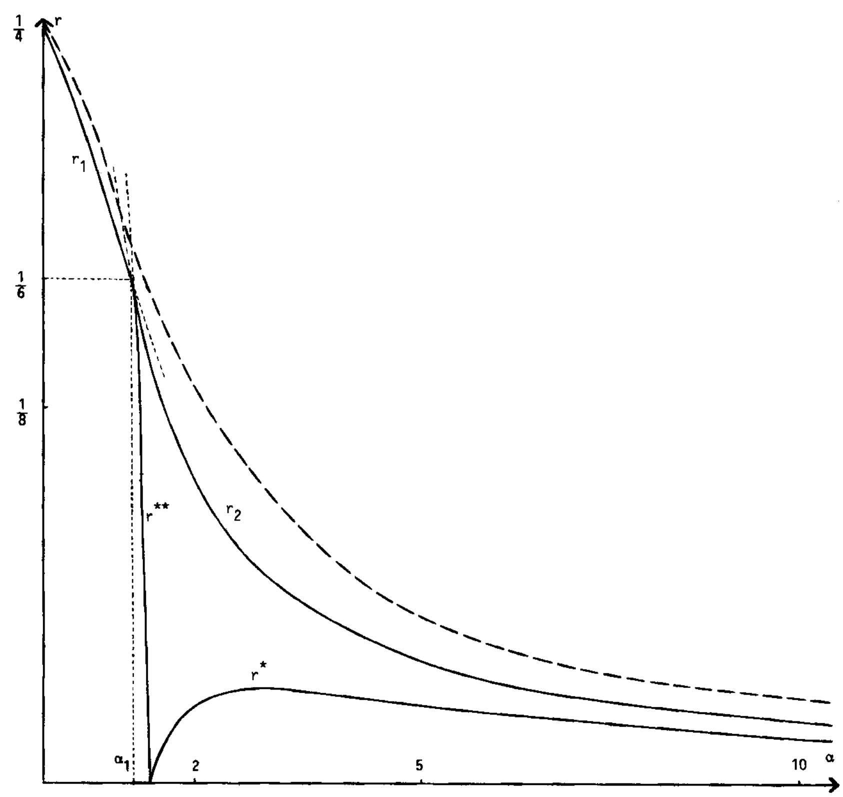

Figure 6. MFTCD scheme: theoretical stability limits ($r_1(\alpha)$ when $\alpha<\alpha_1=\sqrt{(1{\cdot}5)}$, $r_2(\alpha)$ when $\alpha>\alpha_1$) and experimental stability limit (broken line)

The weighted schemes are defined by

$$(\theta+\tfrac{1}{2})v_{i,j}^{n+1}-2\theta v_{i,j}^{n}+(\theta-\tfrac{1}{2})v_{i,j}^{n-1}=\Delta t A_h \bar{v}_{i,j}, \quad \theta\in[0,1], \tag{39}$$

with $\bar{v}_{i,j}$ a linear combination of $v_{i,j}^{n}$, $v_{i,j}^{n-1}$ and $v_{i,j}^{n+1}$ and A_h one of the space difference operators studied in Section 2. These schemes are deduced from two-step schemes used in differential equations, so their stability properties are essentially independent of the space operators.

The weighted scheme given by Zlamal[19] is defined by (39) with

$$\bar{v}_{i,j}=\frac{(1+\theta)^2}{4}v_{i,j}^{n+1}+\frac{1-\theta^2}{2}v_{i,j}^{n}+\frac{(1-\theta)^2}{4}v_{i,j}^{n-1}.$$

This scheme is A_0-stable when $\theta>0$ and may be considered as an optimal weighted scheme.

The stability of 2D LF schemes must be specifically analysed. For instance, the stability results of the 1D LFDF scheme are not valid for 2D problems.[16] We give below the stability analysis of 2D LF schemes generalizing (4) and (5). For these three-level schemes, Fourier analysis yields an amplification matrix, the eigenvalues of which must belong to the unit disc.

The corresponding quadratic eigenvalue equation

$$a_2\nu^2+a_1\nu+a_0=0 \tag{40}$$

is analysed through the Schur–Cohn (SC) lemma.[2,7,18]

SC lemma

The roots of (40) lie inside the unit disc if

$$\delta_1=|a_0|^2-|a_2|^2<0, \tag{41}$$

$$|\bar{a}_0a_1-a_2\bar{a}_1|<|\delta_1|. \tag{42}$$

3.1. LF2 Scheme

The LF2 scheme is defined by

$$\frac{v_{i,j}^{n+1}-v_{i,j}^{n-1}}{2\Delta t}=B_h[\theta v_{i,j}^{n+1}+(1-\theta)v_{i,j}^{n-1}]-c_1D_{0x}v_{i,j}^{n}-c_2D_{0y}v_{i,j}^{n}, \tag{43}$$

where B_h is the five-point centred difference operator for the diffusion term.

From the eigenvalue equation of the associated amplification matrix,

$$\nu^2\{1+4\theta[r_1(1-\cos\varphi_1)+r_2(1-\cos\varphi_2)]\}+2i\nu[\mu_1\sin\varphi_1+\mu_2\sin\varphi_2] \\ +4(1-\theta)[r_1(1-\cos\varphi_1)+r_2(1-\cos\varphi_2)]-1=0, \tag{44}$$

we deduce the following stability result.

Theorem 5

The LF2 scheme (43) is stable if

$$[\mu_1^2+4(1-2\theta)^2r_1^2]^{1/2}+[\mu_2^2+4(1-2\theta)^2r_2^2]^{1/2}+2r_1(1-2\theta)+2r_2(1-2\theta)<1. \tag{45}$$

Proof. We apply (41) and (42) to the quadratic equation (44) (recall that $r_i = K_i \Delta t/h_i^2$ and $\mu_i = c_i \Delta t/h_i$).

$$\text{(i)}\quad |a_0|^2 - |a_2|^2 = -1 + 2[r_1(1-\cos\varphi_1) + r_2(1-\cos\varphi_2)] < 0$$

$$\Leftrightarrow 4\left(r_1 \sin^2\frac{\varphi_1}{2} + r_2 \sin^2\frac{\varphi_2}{2}\right)(1-2\theta) < 1,$$

which is always satisfied when $\theta \geqslant 0{\cdot}5$.

If $\theta < 0{\cdot}5$, we obtain

$$r_1 + r_2 < \frac{1}{4(1-2\theta)}. \tag{46}$$

$$\text{(ii)}\quad |\bar{a}_0 a_1 - a_2 \bar{a}_1| < |\delta_1|$$

yields

$$\Gamma(\varphi_1, \varphi_2) = \mu_1 \sin\varphi_1 + \mu_2 \sin\varphi_2 + 4(1-2\theta)\left(r_1 \sin^2\frac{\varphi_1}{2} + r_2 \sin^2\frac{\varphi_2}{2}\right) < 1 \text{ for } (\varphi_1, \varphi_2) \in [0, \pi]^2.$$

This inequality is different from those analysed in Section 2 and, in particular, cannot be geometrically viewed in a simple way.

$\Gamma(\pi, \pi) < 1$ corresponds to (46) and $\Gamma(0, 0)$ is always verified. Thus we have to study $\max \Gamma(\varphi_1, \varphi_2)$ in the open domain $]0, \pi[\times]0, \pi[$. The system $\Gamma'_{\varphi_1} = 0$, $\Gamma'_{\varphi_2} = 0$ gives one value $(\varphi_{1m}, \varphi_{2m})$ in $]\pi/2, \pi[\times]\pi/2, \pi[$. Prescribing $\Gamma(\varphi_{1m}, \varphi_{2m}) < 1$, we obtain the condition (45) after some algebraic calculations. We can easily verify that (45) implies (46); (45) is therefore the only stability condition for the LF2 scheme. □

3.2. *LF3 scheme*

In the LF3 scheme B_h is applied to a convex combination using the three levels n, $n \pm 1$:

$$\frac{v_{i,j}^{n+1} - v_{i,j}^{n-1}}{2\Delta t} = B_h[\beta v_{i,j}^{n+1} + (1-2\beta) v_{i,j}^n + \beta v_{i,j}^{n-1}] - c_1 D_{0x} v_{i,j}^n - c_2 D_{0y} v_{i,j}^n, \quad \beta \in [0, 0{\cdot}5]. \tag{47}$$

The eigenvalue equation of the associated amplification matrix is given by

$$\begin{aligned} &v^2\{1 + 4\beta[r_1(1-\cos\varphi_1) + r_2(1-\cos\varphi_2)]\} \\ &\quad + 2v\{2(1-2\beta)[r_1(1-\cos\varphi_1) + r_2(1-\cos\varphi_2)] + \mathrm{i}[\mu_1 \sin\varphi_1 + \mu_2 \sin\varphi_2]\} \\ &\quad - 1 + 4\beta[r_1(1-\cos\varphi_1) + r_2(1-\cos\varphi_2)] = 0. \end{aligned} \tag{48}$$

The SC lemma yields the following theorem.

Theorem 6

The LF3 scheme is stable if $\beta > 0{\cdot}25$ and

$$\mu_1 + \mu_2 < \frac{\sqrt{(4\beta - 1)}}{2\beta}. \tag{49}$$

Proof. The proof is straightforward.

$$\text{(i)}\quad \delta_1 = |a_0|^2 - |a_2|^2 = -16\beta[r_1(1-\cos\varphi_1) + r_2(1-\cos\varphi_2)]$$

is always negative.

(ii) $|\bar{a}_0 a_1 - a_2\bar{a}_1| = |-8(1-2\beta)[r_1(1-\cos\varphi_1)+r_2(1-\cos\varphi_2)]$
$+16\mathrm{i}\beta[r_1(1-\cos\varphi_1)+r_2(1-\cos\varphi_2)](\mu_1\sin\varphi_1+\mu_2\sin\varphi_2)| < |\delta_1|$

implies $\beta > 0{\cdot}25$ and

$$4(\mu_1\sin\varphi_1+\mu_2\sin\varphi_2)^2\beta^2 < 4\beta-1, \quad \text{i.e. (49).}$$ □

3.3. LFDF scheme

The LFDF scheme which uses an explicit Du Fort–Frankel approximation for the diffusive term is written as

$$\frac{v_{i,j}^{n+1}-v_{i,j}^{n-1}}{2\Delta t} = K_1\frac{v_{i+1,j}^n+v_{i-1,j}^n-v_{i,j}^{n+1}-v_{i,j}^{n-1}}{h_1^2} - c_1 D_{0x} v_{i,j}^n + K_2\frac{v_{i,j+1}^n+v_{i,j-1}^n-v_{i,j}^{n+1}-v_{i,j}^{n-1}}{h_2^2} - c_2 D_{0y} v_{i,j}^n. \quad (50)$$

For this scheme the stability results were frequently incorrect or incomplete.[13,20] Cushman-Roisin[16] gave the correct result via some rather involved analytical work. The SC lemma supported by geometrical considerations gives the following theorem.

Theorem 7

The LFDF scheme (50) is stable if

$$\left(\frac{\mu_1^2}{r_1}+\frac{\mu_2^2}{r_2}\right)(r_1+r_2) < 1. \quad (51)$$

Proof. The amplification matrix relative to (50) yields

$$(1+2r_1+2r_2)v^2 - 2v(2r_1\cos\varphi_1+2r_2\cos\varphi_2-\mathrm{i}\mu_1\sin\varphi_1-\mathrm{i}\mu_2\sin\varphi_2)-(1-2r_1-2r_2)=0. \quad (52)$$

(i) $|a_0|^2-|a_2|^2=\delta_1=-8(r_1+r_2)<0.$

(ii) $|\bar{a}_0 a_1 - a_2\bar{a}_1| < |\delta_1|$

yields

$$|4r_1\cos\varphi_1+4r_2\cos\varphi_2+2(r_1+r_2)(\mathrm{i}\mu_1\sin\varphi_1+\mathrm{i}\mu_2\sin\varphi_2)| < 4(r_1+r_2),$$

i.e.

$$\left(r_1\frac{\cos\varphi_1}{r_1+r_2}+r_2\frac{\cos\varphi_2}{r_1+r_2}\right)^2+(r_1\sin\varphi_1+r_2\sin\varphi_2)^2<1.$$

The above inequalities may be geometrically interpreted by considering both ellipses with centre 0 and semi-axes $(r_i/(r_1+r_2),\ \mu_i)$ (Figure 7). We necessarily prescribe $\mu_1+\mu_2\leqslant 1$ (vertical semi-axes); the sum of horizontal semi-axes is always equal to unity. Moreover, we assign to the sum of the radii of curvature a value less than unity when $(\varphi_1, \varphi_2)\to(0, 0)$ (see Section 2.1):

$$R_i=\frac{(x_i'^2+y_i'^2)^{3/2}}{x_i''y_i'-x_i'y_i''}\to\mu_i^2\frac{r_1+r_2}{r_i} \quad \text{when } (\varphi_1, \varphi_2)\to(0, 0).$$

Thus we derive the stability condition (51). □

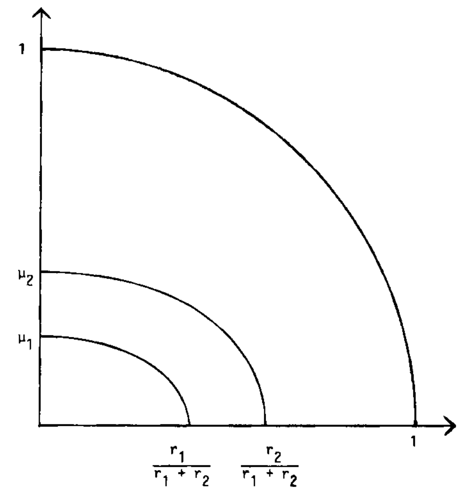

Figure 7. Ellipses illustrating the stability analysis of the LFDF scheme

3.4. LF Schemes: conclusions

The stability properties of the 2D LF schemes do not exactly generalize the results obtained for the 1D schemes in R2.

(a) The stability condition (49) of the LF3 scheme is identical to (63) obtained in R2.
(b) The stability condition (45) of the LF2 scheme is different from (54) obtained in R2; but, for the isotropic problem ($\mu_1=\mu_2$, $r_1=r_2$), (45) may be reduced to the 1D stability condition.
(c) The stability condition (51) of the LFDF scheme is different from (68) in R2, and even with isotropic coefficients, (51) becomes

$$\mu<\tfrac{1}{2}, \quad \text{i.e. } r<1/4\alpha, \tag{53}$$

instead of

$$\mu<1, \quad \text{i.e } r<1/2\alpha,$$

for the 1D scheme.

For isotropic data we plot in Figure 8 the curves $r_{s_i}(\alpha)$:

$r_{s_1}(\alpha)=\dfrac{1}{4[1+\sqrt{(1+\alpha^2)}]}$, stability limit of the LF2 explicit scheme;

$r_{s_2}(\alpha)=1/8\alpha$, stability limit of the LF3 scheme with $\beta=\frac{1}{3}$ (Lees' scheme[21]);

$r_{s_3}(\alpha)=1/4\alpha$, stability limit of the LFDF scheme.

Note that the experimental stability limits are generally very near the r_{s_i}.

These schemes, centred by construction, introduce parasitic oscillations in the numerical solutions when $\alpha>1$. Moreover, recall that the best stability properties of the LFDF scheme when α is not too large are misleading because of the non-consistency of the scheme with (P). This fact requires the use of r significantly smaller than r_{s_3}. Thus the assertion 'LFDF scheme widely used for advective–diffusive problems'[16] must be very cautiously considered.

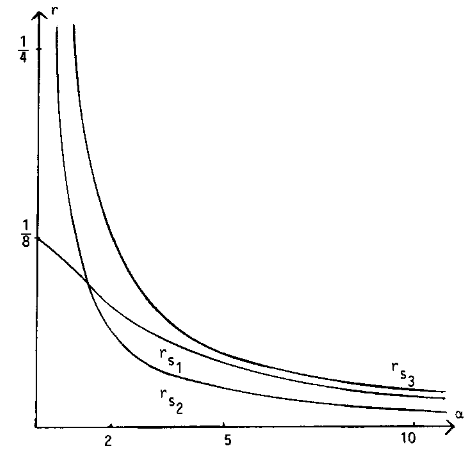

Figure 8. Theoretical stability limits of the LF schemes: LF2 scheme ($r_{s_1}(\alpha)$), LF3 (Lees) scheme ($r_{s_2}(\alpha)$), LFDF scheme ($r_{s_3}(\alpha)$)

4. CONCLUSIONS

This paper is concerned with stability results for two- and three-level 2D schemes which generalize the schemes analysed in our previous papers R1 and R2. We have focused on 2D schemes, the geometrical features of which are easily considered, but the stability properties of 3D schemes are identical to Theorems 1–7 (for some two-level schemes, Hindmarsh *et al.*[3] gave these results in the n-dimensional case).

Table I summarizes the stability results of this paper. For different values of the cell Reynolds number α we give the theoretical (r_t) and experimental (r_e) stability limits of the mesh ratio r when using two- and three-level explicit schemes.

We observe that r_e is either very near or significantly larger than r_t. We have already outlined (remark in Section 2.1) that this fact is dependent on the frequency of unstable modes: near zero, near π or in $]0, \pi[$.

In R1 and R2 we extensively analysed the properties of the solutions of difference schemes. In multidimensional problems the results which only depend on the time discretization are retained: A-stability and positivity.

Concerning the important notions such as artificial viscosity and parabolicity (i.e. monotonicity of A_h) we observe large differences between 1D and 2D schemes. Even for 'product schemes' (five-point in $\mathbb{R}^2$, seven-point in $\mathbb{R}^3$) the 2D properties are not exactly the same as the 1D results, and since we consider nine-point schemes, the properties of the space discretization must be completely analysed.

In particular, the parabolicity, which is essential to the absence of parasitic oscillations, results from the monotonicity of the space difference operator A_h. Some classical results for elliptic problems (e.g. those of Bramble and Hubbard[22,23] and Price[24]) may be applied in this context.

Finally, we must keep in mind that the analysis of the linearized problem (P) is mainly motivated by fundamental quasi-linear problems, e.g. the Navier–Stokes equations. In this case, implicit two-level schemes and weighted three-level schemes yield non-linear systems, and their use, in particular for multidimensional problems, requires important auxiliary work.

Table I. Theoretical (r_t) and experimental (r_e) stability limits of the explicit schemes for some values of the cell Reynolds number α

		α				
		0·25	0·75	2	5	10
MFTCS	r_t	0·243	0·203	0·125	0·02	0·005
	r_e	0·245	0·215	0·135	0·06	0·02
Samarskii	r_t	0·235	0·189	0·107	0·048	0·025
	r_e	0·27	0·275	0·205	0·105	0·055
FTCS	r_t	0·25	0·25	0·061	0·01	0·0025
($\delta = 0$)	r_e	0·255	0·28	0·11	0·021	0·006
FTUS	r_t	0·2	0·143	0·083	0·042	0·023
($\delta = 1$)	r_e	0·22	0·185	0·13	0·075	0·045
MFTCD	r_t	0·243	0·203	0·102	0·041	0·02
	r_e	0·245	0·22	0·13	0·055	0·028
LF2	r_t	0·123	0·111	0·077	0·041	0·0226
($\theta = 0$)	r_e	0·125	0·115	0·078	0·042	0·0228
LFDF	r_t	1	0·333	0·125	0·05	0·025
	r_e	2	0·44	0·135	0·051	0·025

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